MATH 590: QUIZ 8

Name:

1. Let $T : V \to V$ be a linear transformation with $\dim(V) = n$. (i) Define what it means for T to be diagonalizable and (ii) State an equivalent condition to the diagonalizability of T. (4 points)

Solution. (i) T is diagonalizable if there exists a basis consisting of eigenvectors, equivalently, there exists a basis α of V such that $[T]_\alpha^\alpha$ is a diagonal matrix.

(ii) Either $p_T(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$, for distinct $\lambda_i \in F$ **and** $\dim(E_{\lambda_i}) = e_i$ for each $1 \le i \le r$, **or** $p_T(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$, for distinct $\lambda_i \in F$ **and** $\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_r}) = n$.

2. Determine whether or not the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 1 \end{pmatrix}$ is diagonalizable. You must justify your answer. (6 points).

Solution. We have $p_A(x) = \begin{vmatrix} x-2 & 0 & 0 \\ -2 & x-1 & -3 \\ -5 & 0 & x-1 \end{vmatrix} = (x-1)^2(x-2)$, by expanding along the first row. Thus, the algebraic multiplicity of the eigenvalue 1 equals 2. On the other hand, the eigenspace E_1 is the null space of the matrix $A - 1 \cdot I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 5 & 0 & 1 \end{pmatrix}$. This latter matrix row reduces to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, which has rank two and thus, nullity 1. Therefore, the dimension of E_1 is less than 2, so that A is *not* diagonalizable.